No. 8 ON THE PROBLEMS OF SELENODETIC PHOTOGRAMMETRY

by D. W. G. ARTHUR

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AS a technique for dealing with measurements on photographs, photogrammetry is distinguished by its emphasis on stereoscopic measurement and by its insistence on projection of the photographs as cones of perspective rays. In most photogrammetric operations the object is the construction of a three-dimensional model by intersections of these rays.

The importance of stereoscopic measurement to photogrammetrists arises from the principal application of the techniques, namely, the plotting of contoured maps from aerial photographs. The main measuring difficulty is then to ensure that the same point is measured on the various photographs, even when this point lies in terrain which is devoid of all well-defined marks. So long as the photographs contain the vague low-contrast background pattern known as stereoscopic texture, this difficulty is surmounted by stereoscopic methods of measurement.

It does not follow that stereoscopy has the same importance for selenodetic work. Indeed, slight differences of phase, tone, and resolving power may cause the stereoscopic method to break down. In any case, in selenodetic work it is always possible to select from the numerous well-defined craterlets and spots which are present on the Moon's surface, so that stereoscopy is not needed. The other aspects of photogrammetric practice are not unimportant. In the construction of the model by intersecting perspective rays it is a basic principle in photogrammetry to avoid all reference to data external to the photographs. Thus, in principle at least, it is possible to survey the lunar surface in three dimensions without reference to the libration theory, to the orbital theory, or indeed to anything other than the measures of the photographs themselves.

In order to do this, however, it is essential to have a precise knowledge of the so-called inner orientation of the photograph. This is merely the set of data by which the photograph is converted into the associated bundle of perspective rays. In the case of the long-focus astronomical refractor, with its negligible distortions of the image, the inner orientation is defined by the effective focal length and the plate coordinates of the intersection of the optical axis with the focal plane. Again, the narrowness of the field reduces the need for accuracy in the position of this intersection, but the focal length remains important.

Generally speaking the focal lengths of the largest refractors are not known with precision. Our photographic program with the 40-in. refractor of the Yerkes Observatory therefore includes photography of the Pleiades at widely separated temperatures. We hope to establish the focal length as a function of temperature with sufficient accuracy to employ photogrammetric principles to selenodetic problems.

The simplest of the two major selenodetic problems is the determination of the Moon's geometrical figure and this must be based on coordination in three dimensions of a fairly large number of surface points. All attacks on this problem, by Franz, Saunder, Weimer, and Schrutka-Rechtenstamm, have been by way of the libration theory and are therefore not photogrammetric. To be sure, Saunder attempted to apply photogrammetric principles and to derive coordinates from the photographs alone, but his attempt failed. The reasons for this failure will be brought out in what follows.

The analytical formulation for the coordination of the lunar surface from the photographic measurements alone is simple enough. The important step is to realize that such an approach is possible. One such formulation is outlined here. Its only merits are directness and simplicity. The analysis is most easily outlined in terms of 3-vectors and dyadics. Let $\mathbf{r} = (x, y, z)$ be the position vector running from the effective position of the lens to the image of a point on the lunar surface. Then z is the effective focal length of the telescope or camera. The lunar surface points are coordinated in the system (X, Y, Z) which is specified later. In this system let $\mathbf{p} = (X, Y, Z)$ be the position vector of a surface point and let $\mathbf{q} = (E, F, G)$ be the position vector of the camera station. Then the vector running from the camera station to the surface point is $\mathbf{p} - \mathbf{q}$ when resolved in the system (X, Y, Z), and is $\Theta \cdot (\mathbf{p} - \mathbf{q})$ when resolved in the photographic system, Θ being an appropriate rotation dyadic.

Now in a correct solution this vector $\Theta \cdot (\mathbf{p} - \mathbf{q})$ and the photographic vector $\mathbf{r}$ must have the same direction, so our observation equation is:

$$\mathbf{r} \times \Theta \cdot (\mathbf{p} - \mathbf{q}) = 0.$$

This is non-linear in the unknowns Θ, $\mathbf{p}$, $\mathbf{q}$ and is replaced by the corresponding first order form for the corrections, i.e.,

$$\mathbf{r} \times \delta [\Theta \cdot (\mathbf{p} - \mathbf{q})] + \mathbf{r} \times \Theta \cdot (\mathbf{p} - \mathbf{q}) = 0.$$

For convenience we represent the increment in $\Theta \cdot (\mathbf{p} - \mathbf{q})$ due to $\delta\Theta$ as $\omega \times \Theta \cdot (\mathbf{p} - \mathbf{q})$, where ω is a small rotation vector. Subsequently we report the corrections to Θ from

$$\delta\Theta = \omega \times \Theta.$$

With this substitution the equation becomes

$$\mathbf{r} \times [\omega \times \Theta \cdot (\mathbf{p} - \mathbf{q}) + \Theta \cdot (\delta\mathbf{p} - \delta\mathbf{q})] + \mathbf{r} \times \Theta \cdot (\mathbf{p} - \mathbf{q}) = 0.$$

The expansion of this into its scalar equivalents is simple though laborious and need not be described here. Two points should be noted. First, the above is an assertion of the identity of two directions and is therefore equivalent to two, not three, scalar resolutes. For obvious reasons the x – and y – resolutes are chosen. Secondly, it will be found that the absolute values can be worked into the forms

$$x - uz, \qquad y + vz$$

where u and v do not depend on x or y. Thus, the only error-correlation present is that arising from the errors of focal length. In a first essay these can be ignored and the observations regarded as independent.

Thus, if there are m plates and the same n points are observed on each, then there are $2mn$ observation equations. Each plate involves six unknowns and each point three unknowns, so there are $6m + 3n$ unknowns altogether. However, seven of these unknowns are at our disposal to relate the coordinate system (X, Y, Z) to the model. This is most conveniently done by assigning arbitrary complete sets (X, Y, Z) to two of the observed points and by assigning one of X, Y or Z for a third point. In the selenodetic application we can make our coordinate system (X, Y, Z) approximate to the usual selenodetic system by estimating Saunder's (ξ, η, ζ) for these points and using these as the assigned values. We now have $6m + 3n - 7$ unknowns, since the corrections δX, δY, δZ for the assigned values drop out of the equations. For a non-trivial solution the necessary condition is

$$2mn \geqq 6m + 3n - 7.$$

This contains the possibility

$$m = 2, \qquad n = 5,$$

that is, two plates with the same five points observed on each. Indeed, this case corresponds to Fourcade's correspondence theorem, which is well known to photogrammetrists and forms the basis of almost all model-formation in practical photogrammetric procedures.

It is important to realize that it has no practical validity for lunar photographs, a situation which arises from the ultra-narrow fields characteristic of lunar photographs. Photogrammetrists are accustomed to working with cameras with field of 60° to 90°, whereas with lunar photographs we are restricted to fields of 30′. This difference of degree is so extreme that it amounts to a difference in kind. The lunar photograph is virtually an orthographic projection and it can soon be shown that the solution for $m = 2$ is indeterminate for orthographic pictures.

Imagine the two orthographic pictures placed so that the rays are in correspondence, that is, so that each ray from the first picture meets the ray from the corresponding image in the second. Each picture is equivalent to a pencil or beam which is parallel in itself. Now if either pencil is rotated about an axis perpendicular to both pencils, then each ray slides on the corresponding ray and the correspondence is not broken. In other words, the dihedral angle between the picture-planes is indeterminate and the solution fails.

This point was reached by Saunder but he failed to realize that a solution is possible if more than two orthographic pictures are used simultaneously. Considering generally several orthographic pictures at arbitrary and different scales, let these be placed so that the images of one selected point coincide. The pictures are still free to rotate and the unknown elements of mutual orientation are one rotation in the plane of each picture and the dihedral angles between their planes. Taking axes in each plane through the common point we can always write for one pair of pictures i, k,

$$\mu_i(y_i \cos\theta_{ik} - x_i \sin\theta_{ik}) = \mu_k(y_k \cos\theta_{ki} - x_k \sin\theta_{ki}),$$

where μ_i, μ_k are appropriate scale factors while θ_{ik}, θ_{ki} are associated rotations in the two planes. The equation merely asserts that if the projections are at natural scale, then a *y-datum* can be chosen in each such that the y's of all corresponding images are equal. It is this relation which gives orthographic pictures their peculiar character. In principle the above can be applied to the measured images (x_i, y_i), (x_k, y_k) and solved for μ_k, μ_i and θ_{ik}, θ_{ki}. The plates can be paired in $\frac{1}{2}\, m\,(m-1)$ ways so there are $\frac{1}{2}\, m\,(m-1)\,(n-1)$ equations. Assigning one of the μ's there are $(m-1)$ unknowns of this sort and $m\,(m-1)$ unknown θ's. Hence for a solution

$$\tfrac{1}{2} m(m-1)(n-1) \geqq (m+1)(m-1)$$

or,

$$\tfrac{1}{2} m(n-1) \geqq m+1.$$

The dihedral angles have yet to be determined. However, the angle on the plane i between its intersections with the planes j, k, is:

$$a_{i \cdot jk} = \theta_{ij} - \theta_{ik},$$

so these apex-angles are determined with the θ_{ik}. But if the planes are associated in threes, the apex angles also determine the dihedral angles between the planes, since there is nothing more here than the determination of the angles of a spherical triangle when the sides are given.

Hence for a solution we must have $m \geqq 3$, since the triangles cannot exist when $m < 3$. Putting $m = 3$ in the previous inequality we then have

$$3n > 11.$$

That is, the relative orientation of orthographic pictures is determinate when the images of the same four points can be identified on three pictures.

This is undoubtedly the relevant minimum condition for lunar photographs. The relations between the x and y's in the rigorous orthographic case warn us that there are considerably fewer independent bits of information present than may appear at first sight.

The solution is weak, not only because the photographs approximate to orthographic pictures, but also because of the limited range of the optical libration. The rays cross each other very obliquely, so the intersected positions are not well defined. The only way to deal with this, apart from the utmost care in the photography and measurements, is to increase the overdetermination to the limits set by the computing equipment.

As yet we cannot estimate how many plates and points are required to give a useful accuracy, but about twenty plates are available and intuition suggests 200 points or even more. However, we cannot really tell until the normal matrix has been inverted for the first trial calculation. Accuracy at this stage is quite separate from the problem of how many points are required to define the selenoid, since once a satisfactory solution has been achieved, points not used in the solution can be processed with the original observation equation

$$\mathbf{r} \times \Theta \cdot (\mathbf{p} - \mathbf{q}) = 0;$$

leading to 3×3 normal matrices for each point not appearing in the solution.

The second major selenodetic problem is the determination of the Moon's rotation constants. So far the heliometer has been used for this, following, except for minor details, the scheme set out by Bessel. The heliometer is a difficult instrument to use and the reductions are complex. In addition, Bessel's scheme is theoretically unsatisfactory since it links the various observations together by a rather dubious assumption about the smoothed lunar limb. There is no reason to believe that the limb defines a unique point in the Moon's interior, and this hypothesis is *ad hoc* in character. With modern standards of accuracy this approach becomes more and more unsatisfactory.

Now the Moon's rotation, as well as displacing the surface markings with respect to the limb, also displaces them with respect to each other. Thus, the observation equations can be stated in terms of the rectangular coordinates of the points in the plane of the image, without reference to the limb, or at least in such a manner that the limb plays a purely secondary role.

Two approaches are possible. The first is to make use of the spatial coordinates determined as above, assuming these to be related to the usual selenodetic system by a general transformation of rectangular coordinates. In the other approach these coordinates are left free and are determined along with the constants of rotation. In vaguest outline, since this approach has not yet been thought through in detail, the observation equations may be put in the form

$$x = F_1(X, Y, Z, X_0, Y_0, Z_0, I, f),$$
$$y = F_2(X, Y, Z, X_0, Y_0, Z_0, I, f),$$

where (x, y) are the photographic coordinates reduced to the image of the fundamental point as origin, (X, Y, Z) are the rectangular coordinates of the surface point in the selenodetic system but referred to the fundamental point as origin, (X_0, Y_0, Z_0) are the approximate barycentric coordinates of the fundamental point, and I, f represent the unknown rotation elements. The coordinates (x, y) are not very sensitive to errors of (X_0, Y_0, Z_0) and this is just as well since the latter cannot be known with the highest precision. It will be noticed that this scheme replaces coordination of the points with respect to a vague inaccessible origin by coordination with respect to a definite and observable origin, the fundamental point.

The measures will be made on photographs taken with the 40-in. Yerkes refractor, in which star trails are registered for purposes of orientation. These have all been taken by Mr. Elliott Moore. Before starting the evening's work star maps and catalogs are examined to ascertain the gap between the Moon and suitable stars, which must be brighter than the 5th magnitude in order to register a trail. On occasions there are none suitable at the Moon's declination within the limits of the field of the instrument and movement in declination of 2° is permitted to close the time gap between lunar and stellar exposure. Even so, it is sometimes necessary to wait as long as 2 hours in order to get the exposure for the star-trail. During this time the telescope cannot be moved nor used in any way, since there is considerable risk of disturbing the plate. Atmospheric temperatures and pressures are carefully recorded for those plates in order to calculate the differential refraction. The slight change in declination, when present, will be taken into account when computing the orientation of the star-trail. Numerous points on each trail will be measured to establish its orientation with respect to the axes of the measures. The points to be measured fall into two groups, one group near the center of the disk and another group scattered around the limb regions, but not on the limb itself.

Only photographic methods are adequate for this approach and Koziel's doubts about the accuracy of photographs are rejected here. After all, 50 years ago S. A. Saunder obtained relative accuracies of about 0·1 sec in his crater positions and surely we can do better today.